

On two classes of dense 2-generator subgroups in \mathbb{C} .

Kirill Kamalutdinov Andrey Tetenov* Dmitry Vaulin

October 23, 2014

Abstract

We consider dense 2-generator multiplicative subgroups in \mathbb{C} and show that for each point $z \in \mathbb{C}$ the set of limit values for the arguments of the powers of each generator at the point z is either finite or is $[-\pi, \pi]$

MSC classification: Primary 20F38, 28A80

Introduction.

The paper was motivated by some neat examples in fractal geometry [1, 3] and the problems arising in the course of their construction.

It is a well known and widely used fact that if ξ, η are such positive numbers, that $\frac{\log \xi}{\log \eta}$ is irrational, then for any $\lambda > 0$ we can find such sequence (n_k, m_k) , that $\lim_{k \rightarrow \infty} n_k = \infty, \lim_{k \rightarrow \infty} m_k = \infty, \lim_{k \rightarrow \infty} \xi^{n_k} / \eta^{m_k} = \lambda$.

Now suppose that ξ and η are complex numbers. What conditions must be imposed on ξ and η to ensure that such sequence exists for any $\lambda \in \mathbb{C}$? More complicated question is, can we find such ξ, η , that for any $\lambda \in \mathbb{C}$ and for any $\alpha \in (-\pi, \pi)$ there is a sequence (n_k, m_k) , for which $\lim_{k \rightarrow \infty} \xi^{n_k} / \eta^{m_k} = \lambda$, while the values of the arguments of ξ^{n_k}, η^{m_k} converge to a given value α ?

Surprisingly, the answer is yes. This is possible in the case, when a, b are the generators of such group $G = \langle \xi, \eta, \cdot \rangle$, dense in \mathbb{C} , that for each open

*Supported by Russian Foundation of Basic Research project 13-01-00513

set $V \subset \mathbb{C}$ the values of the arguments of first factors ξ^m of the elements $\xi^m \eta^n \in (G \cap V)$ are dense in $[-\pi, \pi]$. We call such G a *group of the second type* and prove that they do exist.

Dense additive and multiplicative groups in \mathbb{C} .

We begin with dense 3-generator lattices in \mathbb{C} .

Theorem 1 *Let $\xi = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$. A group $G = \langle 1, i, \xi, + \rangle$ is dense in \mathbb{C} iff for any integers k, l, m*

$$k\alpha + l\beta + m = 0 \text{ implies } k = l = m = 0, \quad (*)$$

Proof.

Let φ be the canonical homomorphism of G to the factor-group $G' = G/\mathbb{Z}[i]$, where $\mathbb{Z}[i] = \langle 1, i, + \rangle$ is a group of Gaussian integers. G' is a subgroup of the torus $T = \mathbb{C}/\mathbb{Z}[i]$. By (*), the equality $\varphi(m\xi + k + li) = \varphi(m'\xi + k' + l'i)$ holds iff $m = m'$. Therefore all $z_m = \varphi(m\xi + k + li)$ are different in G' . So G' is an infinite cyclic subgroup in T and it has a limit point in T . If a sequence $\{z_{m_k}\}$ converges, the sequence $\{z_{m_{k+1}} - z_{m_k}\}$ converges to 0, so 0 is a limit point both in G' and in G .

For $z \neq 0$ in \mathbb{C} , let L_z be the line $\{tz \mid t \in \mathbb{R}\}$. Notice, that for any bounded subset $D \subset \mathbb{C}$, the set $G \cap L_z \cap D$ is finite.

Indeed, if $z = m\xi + k + li$, and $z' = m'\xi + k' + l'i \in L_z$, then $\frac{m'\alpha + k'}{m\alpha + k} = \frac{m'\beta + l'}{m\beta + l}$. The last equation implies

$$(m'l - l'm)\alpha + (km' - k'm)\beta + (k'l - kl') = 0$$

Then it follows from (*) that $\frac{l'}{l} = \frac{m'}{m} = \frac{k'}{k}$. If $z' \in D$, this ratio is bounded, so $G \cap D \cap L_z$ is finite.

Therefore, for any $\varepsilon > 0$ we can take such $a, b \in G \cap B(0, \varepsilon)$, that $b \notin L_a$. Then the set $\{ka + lb : k, l \in \mathbb{Z}\}$ is an ε -net in \mathbb{C} . This shows that G is dense in \mathbb{C} .

The second part of the proof is rather short.

Suppose $k\alpha + l\beta + m = 0$ and $k \neq 0$. Find such $p_1, p_2, q \in \mathbb{Z}$ that $kp_1 + lp_2 - mq = 0$. If $x = q\alpha + p_1$, $y = q\beta + p_2$ then $kx + ly = kp_1 + lp_2 - qm$.

The last equality means that all the elements of the group G lie on a family of parallel lines $kx + ly = n$, where k, l are fixed and $n \in \mathbb{Z}$. The unit square P intersects no more than $k + l + 1$ of these lines, therefore G is not dense in \mathbb{C} . ■

Applying an affine transformation sending $1, i, \xi$ to u, v, w , we get

Corollary 2 *Let $u, v \in \mathbb{C}$, $Im \frac{u}{v} \neq 0$, $w = \alpha u + \beta v$, $\alpha, \beta \in \mathbb{R}$.*

A group $G = \langle u, v, w, + \rangle$ is dense in \mathbb{C} iff for any integers k, l, m ,

$$k\alpha + l\beta + m = 0 \text{ implies } k = l = m = 0 \quad \blacksquare$$

In the case $w = 1$ we can apply the map $f(z) = e^{2\pi iz}$ to obtain

Corollary 3 *Let $u, v \in \mathbb{C}$, $Im \frac{u}{v} \neq 0$, $\alpha u + \beta v = 1$, $\alpha, \beta \in \mathbb{R}$, $\xi = e^{2\pi i u}$,*

$\eta = e^{2\pi i v}$. A group $G = \langle \xi, \eta, \cdot \rangle$ is dense in \mathbb{C} iff for any integers k, l, m ,

$$k\alpha + l\beta + m = 0 \text{ implies } k = l = m = 0 \quad \blacksquare$$

Then $G = \langle \xi, \eta, \cdot \rangle$ is called a *dense 2-generator multiplicative group* in \mathbb{C} .

We will consider a group $G = \langle \xi, \eta, \cdot \rangle$ along with its additive counterpart, $\hat{G} = \langle u, v, 1, + \rangle$, where $\xi = e^{2\pi i u}$, $\eta = e^{2\pi i v}$.

Notice that if $G = \langle \xi, \eta, \cdot \rangle$ is dense in \mathbb{C} , the formula $\psi(\xi^m \eta^n) = \left(\frac{\xi}{|\xi|} \right)^m$

defines a homomorphism ψ of a group $G = \langle \xi, \eta, \cdot \rangle$ to the unit circle $S^1 \subset \mathbb{C}$. Put

$$H_G = \bigcap_{\varepsilon > 0} \overline{\psi(B(1, \varepsilon) \cap G)}$$

In other words, H_G is the set of limit points of all those sequences $\{e^{in_k \arg(\xi)}\}$, for which $\{n_k\}$ is the first coordinate projection of such sequence $\{(n_k, m_k)\}$, that $\lim_{k \rightarrow \infty} \xi^{n_k} \eta^{m_k} = 1$.

The set H_G is a closed subset of the unit circle S^1 , and it is a subgroup of the group S^1 , so it is either finite cyclic, i.e. $H_G = \{e^{2k\pi i/n}\}$ for some $n \in \mathbb{N}$, or it is infinite and dense in S^1 and therefore $H_G = S^1$.

Definition 4 *A dense 2-generator subgroup G is called the group of the first type, if H_G is finite, and the group of the second type, if $H_G = S^1$.*

Lemma 5 *G is of the second type iff for some $\alpha \notin \mathbb{Q}$, $e^{2\pi i \alpha} \in H_G$. ■*

Remark. Let $\xi = e^{2\pi i u}, \eta = e^{2\pi i v}$ are the generators of the group G of the second type and $e^{2\pi i \alpha} \in H_G$ for some irrational α . From the point of view of the group G' this means, that there is such sequence $\{(p_n, r_n, s_n)\}$ in \mathbb{Z}^3 , that $\lim_{n \rightarrow \infty} (r_n u + s_n v) = 1$ and $\lim_{n \rightarrow \infty} r_n Re(u) - p_n = \alpha$.

In other words, $\lim_{n \rightarrow \infty} \{r_n Re(u)\} = \alpha$ (where $\{x\}$ stands for fractional part of x).

Corollary 6 *If $G = \langle \xi, \eta, \cdot \rangle$ is of the second type, then for any open subset $V \in \mathbb{C}$ the set $\psi(V \cap G)$ is dense in S^1 .*

Proof Take $\xi^m \eta^n \in V$. Then for some $\varepsilon > 0$, $\xi^{-m} \eta^{-n} V \supset B(1, \varepsilon)$. Since $\psi(B(1, \varepsilon))$ is dense in S^1 , the same is true for $\psi(V)$. ■

The groups of the second type have a significant geometric property:

Theorem 7 *If the group G with generators $\xi = re^{i\alpha}, \eta = Re^{i\beta}$ is of the second type, then for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ there is such sequence $\{(n_k, m_k)\}$ that $\lim_{k \rightarrow \infty} \frac{z_1 \xi^{n_k}}{z_2 \eta^{m_k}} = 1$, $\lim_{k \rightarrow \infty} e^{in_k \alpha} = e^{-i \arg(z_1)}$, $\lim_{k \rightarrow \infty} e^{im_k \beta} = e^{-i \arg(z_2)}$.*

The proof directly follows from the previous corollary. ■

The last theorem means, that if G is of the second type, then given any two orbits $\{z_1 \xi^n\}, \{z_2 \eta^m\}$ and any ray $L_\theta : \arg z = \theta$, there are two subsequences $\{z_1 \xi^{n_k}\}, \{z_2 \eta^{m_k}\}$ for which $\lim_{k \rightarrow \infty} \frac{z_1 \xi^{n_k}}{z_2 \eta^{m_k}} = 1$, while both projections $\left(\frac{z_1 \xi^{n_k}}{|z_1 \xi^{n_k}|} \right), \left(\frac{z_2 \eta^{m_k}}{|z_2 \eta^{m_k}|} \right)$ approach $e^{i\theta}$.

Examples of the groups of first and second type.

For the groups of the first type, we have the following sufficient condition:

Theorem 8 *Suppose $G = \langle \xi, \eta, \cdot \rangle$ is dense in \mathbb{C} , $\xi = e^{2\pi i u}, \eta = e^{2\pi i v}$. If*

$$\frac{\operatorname{Re} u \operatorname{Im} v}{\operatorname{Re} v \operatorname{Im} u} \in \mathbb{Q},$$

then G is of the first type.

Proof. Put $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose $\frac{v_1 u_2}{u_1 v_2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$. Notice that u and v are not collinear, so $p \neq q$.

Take some $\varepsilon \in (0, 1)$ and suppose $|\xi^m \eta^n - 1| < \varepsilon$, $m, n \in \mathbb{Z}$. Then $|mu + nv - k| < \varepsilon$ for some $k \in \mathbb{Z}$.

Combining the inequalities $|nu_2 + mv_2| < \varepsilon$, $|mu_1 + nv_1 - k| < \varepsilon$, we get

$$\left| m \frac{u_1 v_2 - v_1 u_2}{v_2} - k \right| < \varepsilon \left(1 + \left| \frac{v_1}{v_2} \right| \right), \text{ or } \left| m \frac{q-p}{q} u_1 - k \right| < \varepsilon \left(1 + \left| \frac{v_1}{v_2} \right| \right).$$

Rewriting the last inequality in the form

$$\left| mu_1 - \frac{kq}{q-p} \right| < \varepsilon \left| \frac{q}{q-p} \right| \left(1 + \left| \frac{v_1}{v_2} \right| \right)$$

we see that the order of the group G divides $|p - q|$. ■

To show the existence of a dense group of the second type, we follow Remark to the Lemma 5 and find such $u, v \in \mathbb{C}$ and $\alpha \in (0, 1) \setminus \mathbb{Q}$, that for some integers r_n, s_n , $\lim_{n \rightarrow \infty} (r_n u + s_n v) = 1$ and $\lim_{n \rightarrow \infty} \{r_n \operatorname{Re}(u)\} = \alpha$. We make it three steps.

Step 1. Construct irrational numbers α, β, γ .

We construct such positive numbers α, β and γ with continued fraction representations

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}, \quad \beta = \frac{1}{b_1 + \frac{1}{b_2 + \dots}}, \quad \gamma = \frac{1}{c_1 + \frac{1}{c_2 + \dots}},$$

that the denominators of the convergents for α, β, γ form the sequences $\{q_n\}$, $\{q_n + 1\}$, $\{q_n + 2 + (-1)^n\}$ respectively. So the convergents for α, β, γ will be

$$\frac{p_n}{q_n}, \quad \frac{r_n}{q_n + 1}, \quad \frac{s_n}{q_n + 2 + (-1)^n}$$

Define the sequence $\{q_n\} = (1, 2, 31, 994, \dots)$ by the relations: $q_1 = 1$, $q_2 = 2$, $q_{2n+1} = q_{2n-1} + q_{2n}(q_{2n} + 1)(q_{2n} + 3)$, $q_{2n+2} = q_{2n} + q_{2n+1}(q_{2n+1} + 1)$.

The terms for α are (see [2]) $a_{n+2} = \frac{q_{n+2} - q_n}{q_{n+1}}$,

so $a_{2n+1} = (q_{2n} + 1)(q_{2n} + 3)$, $a_{2n+2} = q_{2n+1} + 1$, $a_1 = q_1 = 1$.

The terms for β are $b_{2n+1} = q_{2n}(q_{2n} + 3)$, $b_{2n+2} = q_{2n+1}$, $b_1 = 2$.

Then the terms for γ are $c_{2n+1} = q_{2n}(q_{2n} + 1)$, $c_{2n+2} = q_{2n+1}$, $c_1 = 2$.

Finally we get the following sequences a_n, b_n, c_n :

(1, 1, 15, 32, 995 · 997, ...),
(2, 1, 10, 31, 994 · 997, ...),
(2, 2, 6, 32, 994 · 995, ...).

For odd n , we have $\beta = \frac{r_n}{q_n + 1} + \frac{\beta_n}{q_n + 1}$ and $\gamma = \frac{s_n}{q_n + 1} + \frac{\gamma_n}{q_n + 1}$.

Then $\beta_n < \frac{1}{q_n + 1}$ and $\gamma_n < \frac{1}{q_n + 1}$ for any $n \in \mathbb{N}$.

Step 2. Show that β and γ satisfy the condition (*) of Theorem 1.

Suppose $l\beta + m\gamma - k = 0$ for some $l, m, k \in \mathbb{Z}$.

Take such N that for $n > N$, $q_n > 3 \max\{|k|, |m|, |l|\}$.

Rewrite $l\beta + m\gamma - k = lr_n + ms_n - k(q_n + 1) + l\beta_n + m\gamma_n = 0$. Since $l\beta_n + m\gamma_n < \frac{l+m}{q_n + 1} < \frac{2}{3}$, we have $\beta_n = \gamma_n = 0$ for $n > N$. Then for odd $n > N$, we get

$$l \frac{r_n}{q_n + 1} + m \frac{s_n}{q_n + 1} = k \quad \text{and} \quad l \frac{r_{n+1}}{q_{n+1} + 1} + m \frac{s_{n+1}}{q_{n+1} + 3} = k$$

$$\text{Therefore } l \left(\frac{r_n}{q_n + 1} - \frac{r_{n+1}}{q_{n+1} + 1} \right) + m \left(\frac{s_n}{q_n + 1} - \frac{s_{n+1}}{q_{n+1} + 3} \right) = 0.$$

Observing that $r_n(q_{n+1} + 1) - r_{n+1}(q_n + 1) = 1$ and $s_n(q_{n+1} + 3) - s_{n+1}(q_n + 1) = 1$, we get that for any odd $n > N$, $(l + m)q_{n+1} + 3l + m = 0$. This is possible only if $l = m = 0$, so k is also 0.

Step 3. Find such $u, v \in \mathbb{C}$ that $\beta u + \gamma v = 1$, $Re(\beta u) = \alpha$ to construct the group G .

To get desired u, v , take $h > 0$ and define $u = \frac{\alpha + ih}{\beta}$, $v = \frac{1 - \alpha - ih}{\gamma}$.

For any $\varepsilon > 0$ there is such N , that for any $n > N$, $|q_n \alpha - p_n| < \varepsilon$, $|(q_n + 1)\beta - r_n| < \varepsilon$ and $|(q_n + 2 + (-1)^n)\gamma - s_n| < \varepsilon$.

For any odd $n > N$, the sum of the second and third inequalities, multiplied by $|u|, |v|$ respectively, gives $|(q_n + 1) - (r_n u + s_n v)| < \varepsilon(|u| + |v|)$.

From $|(q_n + 1)\alpha - r_n Re(u)| < \varepsilon|u|$ and $|(q_n + 1)\alpha - (p_n + \alpha)| < \varepsilon$ we get $|r_n Re(u) - p_n - \alpha| < \varepsilon(|u| + 1)$.

Using that $0 < \alpha < 1$, and p_n is an integer, for $n = 2k + 1$ we have $\lim_{k \rightarrow \infty} \{r_{2k+1} Re(u)\} = \alpha$ (where $\{x\}$ stands for fractional part of x), while $\lim_{k \rightarrow \infty} (r_{2k+1}u + s_{2k+1}v) = 1$.

Consider the group $G = \langle \xi, \eta, \cdot \rangle$ generated by $\xi = e^{2i\pi u}$ and $\eta = e^{2i\pi v}$. By Corollary 3, G is dense in \mathbb{C} . Since

$$\lim_{k \rightarrow \infty} e^{ir_{2k+1} Re(u)} = e^{2\pi i \alpha},$$

where α is irrational, by Lemma 5 the group G is of the second type.

References

- [1] Aseev V. V., Tetenov A. V., Kravchenko A. S., On Self-Similar Jordan Curves on the Plane, Siberian Mathematical Journal, May/Jun 2003, Vol. 44, Issue 3, p. 379
- [2] Khinchin A. Ya., Continued fractions / Mineola, N.Y.: Dover Publications, 1997.
- [3] Kamalutdinov K.G., Tetenov A.V., Vaulin D.A., Self-Similar Jordan Arcs, which do not satisfy Weak Separation Property, (in preparation)